AN ALGORITHM BASED ON THE DYNAMIC PROGRAMMING PRINCIPLE FOR DETERMINING THE OPTIMAL SYNTHESIS WITHIN THE DYNAMICAL SYSTEMS

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Abstract

This paper proposes a general algorithm for determining the optimal synthesis of the dynamical systems that take part in classes of optimal control problems of Bolza type with hamiltonian diferentiable. This algorithm is based on the dynamic programming principle.

keywords: dynamical systems, Bolza problem, hamiltonian diferentiable, Cauchy problem

1. THE PRELIMINARY CONCEPT

In [1] then is applied the dynamic programming method for solving the optimal Bolza type control problems with Hamiltonian diferentiable, in case of unautonomy, resulting the sufficient conditions of optimality. Based on these conditions a general algorithm for the optimal synthesis calculus of problem like this was elaborated.

As an application it was used the algorithm for solving it the linear–quadric problem without restrictions (linear regulator problem), obtaining the same results like using the algorithm of Riccati matriceal differential equation.

The principal concepts and definitions used in this paper are:

Definition 1.1

It is named (unautonomous) <u>command system</u>, the following:

$\sum = \left(E_0, E_F, U(\cdot,\cdot),\ f(\cdot,\cdot,\cdot), \mathrm{U}\ (\cdot,\cdot) \right)$, where:

1). $E_0, E_F \subset \mathbf{R} \times \mathbf{R}^n$ are disjunct and nonempty sets and represent the initial and final events multitude; in addition, $E_F \subset \overline{E}_0$ (E_0 is dense in E_F);

2). $U(.,.) : E_0 \to \mathrm{P}_0\left(R^m\right)$ is the command restriction multifunction (application with values in nonempty sets) and has the property that its graph, $Y_0 = G(U(\cdot,\cdot)) = \{(t,x,u) | (t,x) \in E_0, u \in U(t,x)\}$ is a relative close set in $E_0 \times R^m$;

3). $f(\cdot,\cdot,\cdot) : Y_0 \to R^m$ is a Peano-Lipschitz parameterized vectorial field, defined on the open set $Y \subset R \times R^n \times R^m$ and $Y_0 \subset Y$;

4). For each $(t_0, x_0) \subset E_0$, admitted command set $\mathrm{U}\ (t_0, x_0)$ in comparison with initial point (t_0, x_0) is one of the sets $\mathrm{U}_m(t_0, x_0)$, $\mathrm{U}_r(t_0, x_0)$ or $\mathrm{U}_{\varphi}(t_0, x_0)$ of all the applications $u(\cdot) : \left[t_0, t_F\left(t_0, x_0, u(\cdot)\right)\right] \to U\left(E_0\right) \subset R^m$ for which the solution $x(.; t_0, x_0, u(.))$ of the Cauchy problem $\frac{dx}{dt} = f(t, x, u(t)), \quad x(t_0) = x_0$

(1.1)

exists, is unique and defined on the whole interval $\left[t_0, t_F\left(t_0, x_0, u(.)\right)\right] \to U\left(E_0\right) \subset R^m$ and verify the conditions:

a). $\left(t_F\left(t_0, x_0, u(\cdot)\right), x_F\left(t_0, x_0, u(\cdot)\right)\right) \in E_F$, where: (1.2)

$x_F\left(t_0, x_0, u(\cdot)\right) = x\left(t_F\left(t_0, x_0, u(\cdot)\right); t_0, x_0, u(\cdot)\right)$;

b). $\left(t, x\left(t; t_0, x_0, u(\cdot)\right)\right) \in E_0, (\forall) t \in \left[t_0, t_F\left(t_0, x_0, u(\cdot)\right)\right]$;

(1.3)

c). $u(t) \in U\left(t, x\left(t; t_0, x_0, u(\cdot)\right)\right), (\forall) t \in \left[t_0, t_F\left(t_0, x_0, u(\cdot)\right)\right]$ (1.4)

The solution $x\left(\cdot; t_0, x_0, u(\cdot)\right)$ is named admitted trajectory for the $u(\cdot)$ command. It is demonstrated that in the conditions of this definition, the admitted commands are equivalent classes determined by the following equivalence relation: $u_1(\cdot) \cong u_2(\cdot)$, if these are applications of the same type (meaning that they are measurable, bounded, ruler riglate and continuous on segments), $t_F\left(t_0, x_0, u_1(\cdot)\right) = t_F\left(t_0, x_0, u_2(\cdot)\right) = t_F$ and $u_1(\cdot) = u_2(\cdot)$ almost everywhere on the interval $\left[t_0, t_F\right]$.

Definition 1.2

It is named <u>optimal control problem</u> a pair $\left(\sum, C(\cdot, \cdot; \cdot)\right)$, where $\sum = \left(E_0, E_F, U(\cdot, \cdot); f(\cdot, \cdot, \cdot), \mathrm{U}(\cdot, \cdot)\right)$ is a unautonomuos command system (definition 1.1), $C(\cdot, \cdot; \cdot) : G\left(\mathrm{U}(\cdot, \cdot)\right) \to R$ is a function defined on the graph of the admitted command multifunction, $G\left(\mathrm{U}(\cdot, \cdot)\right) = \left\{t_0, x_0, u(\cdot) \middle| (t_0, x_0) \in E_0, u(\cdot) \in \mathrm{U}(t_0, x_0)\right\}$ and represent the cost functional associated to the $\sum$ command system.

For every $(t_0, x_0) \in E_0$, is named <u>optimal command</u> for the $\left(\sum, C(\cdot, \cdot; \cdot)\right)$ problem, relative to the initial point (t_0, x_0) an admitted command $\tilde{u}(\cdot) \in \mathrm{U}(t_0, x_0)$, which verifies the relation: $C\left(t_0, x_0; \tilde{u}(\cdot)\right) = \min\left\{C\left(t_0, x_0; u(\cdot)\right) \middle| u(\cdot) \in \mathrm{U}(t_0, x_0)\right\}$, (1.5)

and the solution $x\left(\cdot; t_0, x_0, \tilde{u}(\cdot)\right)$ is named optimal trajectory relative to the (t_0, x_0). According to this definition, an optimal control problem represents a family of minimization problems for all functionals $C\left(t_0, x_0; \cdot\right) : \mathrm{U}\left(t_0, x_0\right) \to R$, when $(t_0, x_0) \in E_0$ [2], [3].

Dynamic programming method, first presupposes the solving of an infinite-dimensional minimization problem of the functional $C\left(t_0, x_0; \cdot\right) : \mathrm{U}\left(t_0, x_0\right) \to R$ therefore determining an optimal command $\tilde{u}_{t_0, x_0}(\cdot) \in \mathrm{U}(t_0, x_0)$ for every $(t_0, x_0) \in E_0$ and then solving the finite-dimensional minimization problem:

$$W\left(\tilde{t}_0, \tilde{x}_0\right) = \min\left\{W\left(t_0, x_0\right); \left(t_0, x_0\right) \in E_0\right\},$$

(1.6)

where: $W\left(t_0, x_0\right) = C\left(t_0, x_0; \tilde{u}_{t_0, x_0}(\cdot)\right)$.

The complete solution of an optimal control problem is the determination, for each $(t_0, x_0) \in E_0$, of an optimal command according to $\tilde{u}_{t_0, x_0}(\cdot) \in \mathrm{U}\ (t_0, x_0)$, therefore the determination of an optimal command selection for admitted commands multifunction $\mathrm{U}\ (\cdot,\cdot)$.

Definition 1.3

It is named <u>optimal synthesis</u> of the optimal control problem $(\Sigma, C(\cdot,\cdot;\cdot))$, an application $v(\cdot,\cdot): E_0 \to U(E_0)$ with the property that for every $(t_0, x_0) \in E_0$, the Cauchy problem

$$\frac{dx}{dt} = f(t, x, v(t, x)), x(t_0) = x_0 \tag{1.7}$$

admit the solution $\tilde{x}(\cdot; t_0, x_0): [t_0, \tilde{t}_F(t_0, x_0)] \to R^n$, which verifies the integral equation:

$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds,\ t \in I(t_0, x_0)$, so that $\tilde{u}_{t_0, x_0}(\cdot)$ defined by

$$\tilde{u}_{t_0, x_0}(t) = v(t, \tilde{x}(t; t_0, x_0)), \quad t \in [t_0, \tilde{t}_F(t_0, x_0)], \tag{1.8}$$

is an optimal command for the $(\Sigma, C(\cdot,\cdot;\cdot))$ problem with respect to (t_0, x_0).

Definition 1.4

It is named <u>Bolza optimal control problem</u> an optimal control problem $(\Sigma, C(\cdot,\cdot;\cdot))$, for which exist $g(\cdot,\cdot): E_F \to R$ inferior semi-continuous and $f_0(\cdot,\cdot,\cdot): Y \to R$, continuous with reference to arguments ensemble and local-lipschitzian with reference to the second argument on the open set *Y* so that the cost functional is:

$$C(t_0, x_0; u(\cdot)) = g(t_F(t_0, x_0, u(\cdot)), x_F(t_0, x_0, u(\cdot))) + \\ + \int_{t_0}^{t_F(t_0, x_0; u(\cdot))} f_0(t, x(t; t_0, x_0, u(\cdot)), u(t))dt,\ (t_0, x_0) \in E_0, u(\cdot) \in \mathrm{U}\ (t_0, x_0) \tag{1.9}$$

The Bolza optimal control problem is $(B) = (\Sigma,, g(\cdot,\cdot),\ f_0(\cdot,\cdot,\cdot))$.

An essential role in dynamic programming method presented in this paper is the value function of an optimal control problem, defined by:

Definition 1.5

It is named <u>value function</u> of the Bolza optimal control problem $(B) = (\Sigma,, g(\cdot,\cdot),\ f_0(\cdot,\cdot,\cdot))$, the function $W(\cdot,\cdot): E = E_0 \cup E_F \to R$, defined by:

$$W(t_0,x_0)=\begin{cases} g(t_0,x_0) & \text{if } (t_0,x_0)\in E_F \\ \min\{C(t_0,x_0,u(\cdot)) \mid u(\cdot)\in \mathrm{U}\ (t_0,x_0)\} & \text{if } (t_0,x_0)\in E_0 \end{cases} \tag{1.10}$$

where $C(\cdot,\cdot;\cdot)$ is the cost functional (1.9).

The idea for using the value function appears for the first time at Carathéodory, but that is used for solving the variational calculus problems. The method was fundamented by Bellman in "***Dynamic Programming"*** edited in 1957 by Princetown University. He named it **dynamic programming method** and he used it for solving a larger class of optimization problems, named dynamic problems.

2. ALGORYTHM FOR THE OPTIMAL SYNTESIS CALCULUS OF THE BOLZA PROBLEMS WITH DIFFERENTIALE HAMILTONIAN

The algorithm is used for the determination of the optimal synthesis $v(\cdot,\cdot):E_0\to U(E_0)$ (definition 1.3.) for the Bolza problem $(B_m)=(\Sigma_m, g(\cdot,\cdot),\ f_m(\cdot,\cdot,\cdot))$ on the command system $\Sigma_m=(E_0,E_F,U(\cdot,\cdot),\ f(\cdot,\cdot,\cdot),\mathrm{U}_m(\cdot,\cdot))$, where E_0 and E_F are:

$E_0=(T_0,T)\times R^n, E_F=\{T\}\times X_F, X_F\subset R^n$ opened, $T\in \mathbf{R}, T_0\in(-\infty,T)$.

Step I The following presuppositions are verified:

(I.1)

$E_0=(T_0,T)\times R^n, E_F=\{T\}\times X_F, X_F\subset R^n$ opened, $T\in \mathbf{R}, T_0\in(-\infty,T)$;

(I.2) Set $G(U(\cdot,\cdot))=\{(t,x,u)\mid (t,x)\in E_0, u\in U(t,x)\}$ is the intersection of an closed set from $R\times R^n\times R^m$ with $E_0\times R^m$;

(I.3) The applications $f(\cdot,\cdot,\cdot)$ and $f_0(\cdot,\cdot)$ are continuous in all arguments and locally lipschitz in correspondence with the second argument on an opened set *Y* that contains $Y_0=G(U(\cdot,\cdot))$;

(I.4) The function $g(T,\cdot)=g(\cdot):X_F\to R$ is a C^2 class.

Step II The Bolza problem pseudoHamiltonian is defined by,

$\mathrm{H}(t,x,p,u)=\langle p,f(t,x,u)\rangle+f_0(t,x,u)$ for $(t,x,u)\in Y_0, p\in R^n$ and $H(t,x,p)=\min\limits_{u\in U(t,x)}\mathrm{H}(t,x,p,u)$, where $(t,x,p)\in A_0\subset E\times R^n$, A_0 is the set of all points $((t,x),p)\in E_0\times R^n$ for which the function $\mathrm{H}\ (t,x,p,u)$ reaches its minimum on the set $\{U(t,x)\mid (t,x)\in E_0, p\in R^n\}$ the multifunction is built up:

$\{\hat{U}(t,x,p)=\{u\in U(t,x)\mid \mathrm{H}\ (t,x,p,u)=H(t,x,p)\}$ and verifies the presuppositions:

(II.1) For every $s\in X_F,(T,s,Dg(s))\in A_F$ and every $(t,x,p)\in A$, the sections

$A_{t,p}=\left\{x\in R^n \middle| (t,x,p)\in A\right\}$ and $A_{t,x}=\left\{p\in R^n \middle| (t,x,p)\in A\right\}$ are opened in R^n, and $H(t,\cdot,\cdot)$ is continuous and C^2 class is proportional with second and third arguments (*x* and *p*) .

(II.2) The $\hat{U}(\cdot,\cdot,\cdot)$ multifunction is locally bounded (every point $(t,x,p)\in A_0$ admits a bounded neighborhood $\tilde{A}_0\subset A_0$ so that the set $\cup\left\{\hat{U}(r,y,q)\middle|(r,y,q)\in\tilde{A}_0\right\}$ is bounded).

Step III We consider the associated Hamiltonian system, with terminal conditions:

$$\begin{cases}\dfrac{dx}{dt}=D_3H(t,x,p),\ \ x(T)=s\in X_F\subset R^n,\\[2ex] \dfrac{dp}{dt}=-D_2H(t,x,p),\ \ p(T)=Dg(s)\end{cases}$$

and the maximal solution is determined (2.1)

$X^*(\cdot,s)=\left(X(\cdot,s),P(\cdot,s)\right):\left(t(s),T\right]\to R^n\times R^n$, for any $s\in X_F$.

Step IV *For each* $t\in\left(T_0,T\right]$ *it can be determined the section of* D *sets by* t,

$D_t=\left\{s\in X_F\middle|t\in I(s)=\left(t(s),T\right]\right\}\subset X_F$ (obviously, $D_T=X_F$) and all opened subsets $\tilde{D}_t\subset D_t$, maximal in ratio with inclusion for which the restriction $X(t,\cdot):\tilde{D}_t\to X\left(t,\tilde{D}_t\right)$ is reversible. Let $S(t,\cdot)=\left(X(t,\cdot)\Big|_{\tilde{D}_t}\right)^{-1}$. For every selection $t\mapsto\tilde{D}_t$ of sets like this, with property: $s\in\tilde{D}_t\Rightarrow s\in\tilde{D}_r,(\forall)r\in[t,T)$, is can be determined $t_1=\inf\left\{t\le T\middle|\tilde{D}_t\ne\varnothing\right\}$ and there are retained only the selections for which $t_1<T$. The next steps of the algorithm are made for every selection like these.

Step V *A selection* $t\mapsto\tilde{D}_t$ *is chosen, with properties from step IV and it verifies the next presupposition:* $S(\cdot,\cdot)$ *application, determined at the IV^th^ step:* $S(t,\cdot)=\left(X(t,\cdot)\Big|_{\tilde{D}_t}\right)^{-1}$ *is differentiable on the set* $\tilde{E}_0\subset E_0$, *where:* $\tilde{E}_0=\left\{(t,x)\in E\middle|t\in(t_1,T),x\in X\left(t,\tilde{D}_t\right)\right\}$.

Step VI It is determined an application $v(\cdot,\cdot):\tilde{E}_0\to U\left(\tilde{E}_0\right)$, measurable, which verifies the condition:

$$v(t,x)\in\hat{U}\left(t,x,P\left(S(t,x)\right)\right),(\forall)(t,x)\in\tilde{E}_0 \qquad (2.2)$$

and is retained as an optimal synthesis for the problem $\left(\tilde{B}_m\right)$ obtained from the given problem $\left(B_m\right)$ by replacing set E_0 in presupposition (I.1) with set $\tilde{E}_0$ obtained in the 5th step. For each $(t_0,x_0)\in\tilde{E}_0$ is retained $\tilde{x}(\cdot;x_0)=X\left(\cdot;S(t_0,x_0)\right)$ as optimal trajectory and

$\tilde{u}_{t_0, x_0}(t) = v(t; \tilde{x}(t; t_0, x_0)), t \in [t_0, T]$, as optimal command for the $(\tilde{B}_m)$ problem, relative at the initial point (t_0, x_0).

Step VII For each $s \in X_F$ it can be determined:

$$X^0(t,s) = g(s) + \int_T^t \left[\left\langle P(r,s), D_3 H(r, X(r,s), P(r,s)) \right\rangle - H(r, X(r,s), P(r,s)) \right] \mathrm{dr}$$

It is calculated the value function of the problem $(\tilde{B}_m)$, respectively $W(t,x) = X^0(t, S(t,x))$, where $(t,x) \in \tilde{E}_0 \cup E_0$. The 5[th], 6[th] and 7[th] steps are returned for every selection $t \mapsto \tilde{D}_t$ determinated at the 6[th] step.

The algorithm presented in this paper is very general; it is preponderantly theoretical and it can be utilized to solve the Bolza optimal control problems if their hypotheses are consistent with the ones settled in this paper.

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