AN ALGORITHM FOR SOLVING OF THE QADRATIC LINEAR PROBLEMS WITH RESTRICTIONS USYNG THE DYNAMIC PROGRAMMING METHOD

Valentin GÂRBAN*

Abstract

This paper proposes using of the general algorithm presented in [11] for solving the linear–quadric problem without restrictions (linear regulator problem), obtaining the same results like using the algorithm of Riccati matriceal differential equation.

The method for solving the quadric linear problem without restriction, is based on the optimal synthesis algorithm of the Bolza problem with differentiable Hamiltonian and terminal fixed time, presented in [11].

The principal concepts and definitions used in this paper are:

Definition 1

The quadric linear problem without restrictions is a Bolza problem, with terminal fixed time, $(B)=\left(\Sigma, g(\cdot,\cdot),\ f_0(\cdot,\cdot,\cdot)\right)$ on the command system

$\Sigma=\left(E_0, E_F, U(\cdot,\cdot),\ f(\cdot,\cdot,\cdot), \mathrm{U}\ (\cdot,\cdot)\right)$, with the following elements, with mentioned properties:

$$\begin{cases} E_0=(-\infty,T)\times\mathbf{R}^n, E_F=\{T\}\times\mathbf{R}^n; U(t,x)=\mathbf{R}^m,(\forall)(t,x)\in E_0\cup E_F; \\ f(t,x,u)=A(t)\cdot x+B(t)\cdot u;\ f_0(t,x,u)=\frac{1}{2}\langle F(t)\cdot x,x\rangle+\langle G(t)\cdot u,x\rangle+ \\ +\frac{1}{2}\langle H(t)\cdot u,u\rangle,(\forall)(t,x,u)\in Y=(-\infty,T]\times\mathbf{R}^n\times\mathbf{R}^m; \\ g(T,x)=g(x)=\frac{1}{2}\langle Cx,x\rangle,(\forall)x\in\mathbf{R}^n \end{cases} \quad (1)$$

* Associate Professor, Ph.D., TITU MAIORESCU UNIVERSITY, Bucharest, România

where: a). $T \in \mathbf{R}, C \in \mathrm{L}\left(\mathbf{R}^n, \mathbf{R}^n\right)$;

b). application with matrix values:

$$A(\cdot), F(\cdot) : (-\infty, T] \to \mathrm{L}\left(\mathbf{R}^n, \mathbf{R}^n\right); H(\cdot) : (-\infty, T] \to \mathrm{L}\left(\mathbf{R}^m, \mathbf{R}^m\right);$$

$$B(\cdot), G(\cdot) : (-\infty, T] \to \mathrm{L}\left(\mathbf{R}^m, \mathbf{R}^n\right); \left(\mathrm{L}\left(\mathbf{R}^m, \mathbf{R}^n\right) \approx \mathrm{M}_{n,m}(\mathbf{R})\right);$$

are continuous and for very $t \in (-\infty, T]$, matrixes $F(t), C \in \mathrm{L}\left(\mathbf{R}^n, \mathbf{R}^n\right)$ and $H(t) \in \mathrm{L}\left(\mathbf{R}^m, \mathbf{R}^m\right)$ are symmetric, and $H(t)$ is strictly positive defined.

<u>Step I</u> With these proprieties, results that the 1st step supposition are verified.

<u>Step II</u> The pseudohamiltonian $\mathrm{H}(t,x,p,u) = \langle p, f(t,x,u)\rangle + f_0(t,x,u)$ in that case is: $\mathrm{H}(t,x,p,u) = \frac{1}{2}\langle H(t)\cdot u, u\rangle + \langle G^*(t)\cdot x + B^*(t)\cdot p, u\rangle +$

$$+\frac{1}{2}\langle F(t)\cdot x, x\rangle + \langle A(t)\cdot x, p\rangle, (\forall)(t,x,p,u) \in (-\infty, T] \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^m, \quad (2)$$

where $G^*(t)$ and $B^*(t)$ are the transposes of the matrix $G(t)$, and $H(t)$. The Hamiltonian of the problem is: $H(t,x,p) = \min_{u \in U(t,x)} \mathrm{H}(t,x,p,u) = \min_{u \in \mathbf{R}^m} \mathrm{H}(t,x,p,u)$. In that case, for every $(t,x,p) \in (-\infty, T] \times \mathbf{R}^n \times \mathbf{R}^n$, the function $\mathrm{H}(t,x,p,\cdot) : \mathbf{R}^n \to \mathbf{R}$ has the Hessian $D_4^2\mathrm{H}(t,x,p,u) = \frac{\partial^2}{\partial u^2}\mathrm{H}(t,x,p,u) = H(t)$ and, in order with the supposition, $H(t)$ is strictly positive defined for every $t \in (-\infty, T]$. ($\Leftrightarrow \langle H(t)u, u\rangle > 0, (\forall) u \in \mathbf{R}^m \setminus \{0_m\}$ and it can be demonstrated that is not singular so that is inversable for every $t \in (-\infty, T]$). It is known that a function is convex on a set only if its Hessian is positive defined on that set [4] and every point of relative minim of a convex function on a convex set, is the absolute minim of the function on that set [5]. All that results take place only in our problem context for the function $\mathrm{H}(t,x,p,\cdot) : \mathbf{R}^m \to \mathbf{R}$ and, the minim point is determined by the equation $D_4\mathrm{H}(t,x,p,u) = 0$. Considering the expression for H (2), it results:

$$H(t)u + G^*(t)x + B^*(t)p = 0 \quad (3)$$

Then, the multifunction $\hat{U}(\cdot,\cdot,\cdot)$ of the minim point is a function type and is: $\hat{U}(t,x,p) = \left\{\hat{u}(t,x,p) \middle| (t,x,p) \in A = (-\infty, T] \times \mathbf{R}^n \times \mathbf{R}^n\right\}$, where:

$$\hat{u}(t,x,p) = -\left(H(t)\right)^{-1}\left(G^*(t)x + B^*(t)p\right). \quad (4)$$

Replacing in the expression of $\mathrm{H}\ (t,x,p,u)$ there can be obtained the quadric linear Hamiltonian of the problem, without restrictions:

$$H(t,x,p)=\mathrm{H}\ \big(t,x,p,\hat{u}(t,x,p)\big)=\langle A(t)\,x,p\rangle+\frac{1}{2}\langle F(t)\,x,x\rangle-$$

$$-\frac{1}{2}\Big\langle G^{*}(t)x+B^{*}(t)\,p,\big(H(t)\big)^{-1}\big(G^{*}(t)x+B^{*}(t)\,p\big)\Big\rangle,\ \text{or} \qquad (5)$$

$$H(t,x,p)=\langle A(t)x,p\rangle+\frac{1}{2}\langle F(t)x,x\rangle-\frac{1}{2}\langle H(t)\hat{u}(t,x,p),\hat{u}(t,x,p)\rangle, \quad (5')$$

where $\hat{u}(t,x,p)$ is defined in (4). From the Hamiltonian expression $H(t,x,p)$ and its constructive elements properties, specified by presuppositions, the results take place in 2nd step presuppositions.

Step III The Hamiltonian system, associated with final conditions:
$\begin{cases}\dfrac{dx}{dt}=D_3H(t,x,p),\ \ x(T)=s\in X_F,\\ \dfrac{dp}{dt}=-D_2H(t,x,p),\ \ p(T)=Dg(s)\end{cases}$, becomes (6) system:

$$\begin{cases}\dfrac{dx}{dt}=\Big(A(t)-B(t)\big(H(t)\big)^{-1}G^{*}(t)\Big)x-B(t)\big(H(t)\big)^{-1}B^{*}(t)\,p,\ \ x(T)=s\in\mathbf{R}^{n}\\ \dfrac{dp}{dt}=\Big(G(t)\big(H(t)\big)^{-1}G^{*}(t)-F(t)\Big)x-\Big(A^{*}(t)-G(t)\big(H(t)\big)^{-1}B^{*}(t)\Big)p,p(T)=Cs\end{cases}$$

So that, the Cauchy problem for Hamilton-Jacobi equation, associated with the quadric linear problems without restrictions is defined by a linear differentiable system with continuous applications for the coefficients. Results ([6], [7]) that, for every $s\in\mathbf{R}^{n}$ the system admits a unique global solution: $X^{*}(\cdot;s)=\big(X(\cdot,s),P(\cdot,s)\big):(-\infty,T]\times\mathbf{R}^{n}\times\mathbf{R}^{n}$, C^{1} class and, in addition, has the propriety, that for every $t\in(-\infty,T]$, the partial application $X^{*}(t,\cdot):\mathbf{R}^{n}\to\mathbf{R}^{n}\times\mathbf{R}^{n}$ is linear. In this concrete case, that means: if $\big(X(\cdot),P(\cdot)\big):(-\infty,T]\to\mathrm{L}\ \big(\mathbf{R}^{n},\mathbf{R}^{n}\big)\times\mathrm{L}\ \big(\mathbf{R}^{n},\mathbf{R}^{n}\big)$ is the matriceal solution of the system (6) with terminal solutions $X(T)=I_{\mathbf{R}^{n}}$ (identity application) and $P(T)=C$, (7)

then the solution $X^{*}(\cdot,s)$ of the (6) problem is:

$$X(t,s)=X(t)\cdot s;P(t,s)=P(t)\cdot s,(\forall)(t,s)\in D=(-\infty,T]\times\mathbf{R}^{n}. \quad (8)$$

Step IV From the last relation results for every $t\in(-\infty,T]$, the D_t section of set $D=(-\infty,T]\times\mathbf{R}^{n}$ through *t* is $\mathbf{R}^{n}$ and $X(t,\cdot)=X(t)$ is inversable on an open set $\tilde{D}_t\subset D_t=\mathbf{R}^{n}$ only if $X(t)\in\mathrm{L}\ \big(\mathbf{R}^{n},\mathbf{R}^{n}\big)$ is not singular ($\Leftrightarrow\det\big(X(t)\big)\neq 0$). But, from $X(T)=I_{\mathbf{R}^{n}}$ results that $\det\big(X(T)\big)=1$ and how the determinant function

$\det(\cdot): L\left(\mathbf{R}^n, \mathbf{R}^n\right) \to \mathbf{R}$ is continuous (like $X(\cdot)$), from the inversability conservation theorem on a vicinity [8], it results that exists $\varepsilon > 0$ so that $\det(X(t)) \neq 0$, for every $t \in (T-\varepsilon, T)$. That, $t_1 \in (-\infty, T]$ defined by:

$$t_1 = \inf\left\{t \le T \middle| \det X(r) \neq 0, (\forall) r \in [t,T]\right\} \quad (9)$$

has the propriety that $t_1 < T$ and the only selection of maximal opened sets, $t \mapsto \tilde{D}_t \subset D_t = \mathbf{R}^n$ with properties from the 4th step of the algorithm is the multifunction $t \mapsto \mathbf{R}^n$ defined for $t \in (t_1, T]$. From relation (8) it also results that the inverse $S(t,\cdot) = (X(t,\cdot))^{-1}$ can be written like this form:

$$S(t,x) = (X(t))^{-1} \cdot x, \quad (\forall)(t,x) \in \tilde{E} = \tilde{E}_0 \cup E_F, \tilde{E}_0 = (t_1,T) \times \mathbf{R}^n \quad (10)$$

<u>Step V</u> From the $S(\cdot,\cdot)$ expression (10) and from its own proprieties, it results that is C[1] class in reference with both arguments on set $\tilde{E}_0 = (t_1,T) \times \mathbf{R}^n$, so it is verified the 5th step presupposition of the algorithm presented in [11].

<u>Step VI</u>

Defining the application $v(\cdot,\cdot): E \to U(E) = \mathbf{R}^m$, $E = E_0 \cup E_F$ and using relations (4), (8) and (10), we finally obtain:

$$v(t,x) = \tilde{u}\left(t,x,P(t)(X(t))^{-1}x\right) = -(H(t))^{-1}\left(G^*(t) + B^*(t)P(t)(X(t))^{-1}\right)x \quad (11)$$

We keep the application $v(\cdot,\cdot)$ as optimal synthesis of the problem $(\tilde{B})$, obtained from the initial problem (B) by replacing the set $E_0(-\infty,T) \times \mathbf{R}^n$ with set $\tilde{E}_0 = (t_1,T) \times \mathbf{R}^n$. From its expression and its own constitutive element proprieties it results that $v(\cdot,\cdot)$ is continuous with regard to both arguments and linear in reference with regard to the second argument. For every $(t,x) \in \tilde{E}_0$, we keep the application

$$\tilde{x}(r;t,x) = X(r) \cdot (X(t))^{-1} \cdot x, \quad r \in [t,T] \quad (12)$$

as optimal trajectory for problem $(\tilde{B})$ relative to the initial point (t,x) and application $\tilde{u}_{t,x}(\cdot)$ defined by: $\tilde{u}_{t,x}(r) = v(r,\tilde{x}(r;t,x)), \quad r \in [t,T]$ (13)
as appropriate optimal command. In addition, from previously mentioned properties it results that $\tilde{u}_{t,x}(\cdot)$ is continuous.

<u>Step VII</u> For the calculus of the value function $W(\cdot,\cdot)$ of the problem $(\tilde{B})$, $W(t,x) = X^0(t,S(t,x))$, where $(t,x) \in \tilde{E} = (t_1,T] \times \mathbf{R}^n = \tilde{E}_0 \cup E_F$, it is necessary to compute first the function $X^0(t,s)$, for every $s \in \mathbf{R}^n (= X_F)$. Usualy,

$$X^0(t,s)=g(s)+\int_T^t\left[\left\langle P(r,s),D_3H\left(r,X(r,s),P(r,s)\right)\right\rangle-H\left(r,X(r,s),P(r,s)\right)\right]dr$$

In this case, considering that $\tilde{x}$ and $\tilde{u}$ are the trajectory, respectively the optimal command relative at the initial point (t,x), it results, for $X^0(t,s)$, the following expression:

$$X^0(t,s)=g(s)+\int_t^T f_0\left(r,X(r,s),\hat{u}\left(r,X(r,s),P(r,s)\right)\right)dr \qquad (14)$$

Considering the particular form of the associated Hamiltonian system (6), we can demonstrate that for every solution $\left(x(\cdot),p(\cdot)\right)$ the following relation is true:

$$\frac{d}{dt}\left\langle x(t),p(t)\right\rangle=-2f_0\left(t,x(t),\hat{u}\left(t,x(t),p(t)\right)\right), \qquad (15)$$

where, $\hat{u}(t,x,p)=-\left(H(t)\right)^{-1}\left(G^*(t)x(t)+B^*(t)p(t)\right)$ is explained in (4).

Indeed: $\frac{d}{dt}\left\langle x(t),p(t)\right\rangle=\left\langle\frac{dx(t)}{dt},p(t)\right\rangle+\left\langle x(t),\frac{dp(t)}{dt}\right\rangle\overset{(3.6)}{=}$

$=\left\langle p,\left(A-BH^{-1}G^*\right)x-BH^{-1}B^*p\right\rangle+\left\langle x,\left(GH^{-1}G^*-F\right)x-\left(A^*-GH^{-1}B^*\right)p\right\rangle=$

$=\left\langle p,Ax\right\rangle-\left\langle p,BH^{-1}G^*x\right\rangle-\left\langle p,BH^{-1}B^*p\right\rangle+\left\langle x,GH^{-1}G^*x\right\rangle-\left\langle x,Fx\right\rangle-\left\langle x,A^*p\right\rangle+$

$+\left\langle x,GH^{-1}B^*p\right\rangle.$

where, for simplifying the writing form, we gave up the variable t. Using the definition of the transpose Q^* of a linear application $Q\in L\left(\mathbf{R}^n,\mathbf{R}^n\right)$, $\left\langle Qu,v\right\rangle=\left\langle u,Q^*v\right\rangle$, $(\forall)\ u,v\in\mathbf{R}^n$, it results $\left\langle p,Ax\right\rangle=\left\langle A^*p,x\right\rangle$ and $\left\langle p,BH^{-1}G^*x\right\rangle=$ $=\left\langle\left(BH^{-1}G^*\right)^*p,x\right\rangle=\left\langle G\left(H^{-1}\right)^*B^*p,x\right\rangle=\left\langle GH^{-1}B^*p,x\right\rangle$, where we used the transpose proprieties and H^{-1} is symmetric. Replacing and reducing the terms, it results, successive: $\frac{d}{dt}\left\langle x,p\right\rangle=\left\langle A^*p,x\right\rangle-\left\langle GH^{-1}B^*p,x\right\rangle-\left\langle p,BH^{-1}B^*p\right\rangle+$ $+\left\langle x,GH^{-1}G^*x\right\rangle-\left\langle x,Fx\right\rangle-\left\langle A^*p,x\right\rangle+\left\langle GH^{-1}B^*p,x\right\rangle.$ Finally, it can be obtained:

$$\frac{d}{dt}\left\langle x,p\right\rangle=-\left\langle Fx,x\right\rangle+\left\langle x,GH^{-1}G^*x\right\rangle-\left\langle p,BH^{-1}B^*p\right\rangle.$$

On the other hand, considering the expressions: $f_0(t,x,u)=\frac{1}{2}\left\langle F(t)x,x\right\rangle+$

$+\langle G(t)u,x\rangle+\frac{1}{2}\langle H(t)u,u\rangle$ and $\hat{u}(t,x,p)=-(H(t))^{-1}\left(G^*(t)x+B^*(t)p\right)$, and making the calculus, we can obtain:

$$-2f_0\left(t,x(t),\hat{u}\left(t,x(t),p(t)\right)\right)=-\langle Fx,x\rangle+\left\langle GH^{-1}G^*x,x\right\rangle-$$

$$-\left\langle p,BH^{-1}B^*p\right\rangle,\text{ because }\left\langle GH^{-1}B^*p,x\right\rangle=\left\langle p,\left(GH^{-1}B^*\right)^*x\right\rangle=\left\langle p,BH^{-1}G^*x\right\rangle.$$

So, it takes place the relation: $\frac{d}{dt}\langle x(t),p(t)\rangle=-2f_0\left(t,x(t),\hat{u}\left(t,x(t),p(t)\right)\right)$, (15)

From where: $f_0\left(t,x(t),\hat{u}\left(t,x(t),p(t)\right)\right)=-\frac{1}{2}\frac{d}{dt}\langle x(t),p(t)\rangle$. Replacing in expression (14) the $X^0(t,s)$ and considering the terminal condition $X^0(T,s)=g(T,s)=g(s)$, it results:

$$X^0(t,s)=\frac{1}{2}\langle X(t,s),P(t,s)\rangle=\frac{1}{2}\langle X(t)s,P(t)s\rangle,(\forall)(t,s)\in\tilde{E}=(t_1,T]\times\mathbf{R}^n \quad (16)$$

So that, the value function of the problem $(\tilde{B})$ is given by relation:

$$W(t,x)=X^0\left(t,S(t,x)\right)=\frac{1}{2}\left\langle P(t)\left(X(t)\right)^{-1}x,x\right\rangle,\quad(\forall)(t,x)\in\tilde{E} \quad (17)$$

and $(X(\cdot),P(\cdot))$ is the matriceal solution of the associated Hamiltonian system (6), which verifies the terminal conditions (7). The particular form of the system (6) permits to obtain a representation for the optimal synthesis $v(\cdot,\cdot)$ and the value function $W(\cdot,\cdot)$, as a solution of a well-known Riccati matriceal differential equation associated to the quadric linear problem, defined by (1). This thing is possible with the utilization of the next theorem results.:

Theorem 2

If $(X(\cdot),P(\cdot))$ is the matriceal solution of the Hamiltonian system (6) with terminal conditions (7) associated to the quadric linear problem (1) and t_0 is given by $t_0=\inf\left\{t\le T\middle|\det X(r)\neq 0,(\forall)r\in[t,T]\right\}$, then the application $R(\cdot):(t_0,T]\to L\left(\mathbf{R}^n,\mathbf{R}^n\right)$, is defined : $R(t)=P(t)\left(X(t)\right)^{-1},\ t\in(t_0,T]$, (18)

and is a solution to the Riccati matriceal differential equation associated to the same problem, (1):

$$\frac{dR(t)}{dt}=R(t)B(t)\left(H(t)\right)^{-1}B^*(t)R(t)-\left(A^*(t)-G(t)\left(H(t)\right)^{-1}B^*(t)\right)R(t)-$$

$$-R(t)\left(A(t)-B(t)\left(H(t)\right)^{-1}G^*(t)\right)+G(t)\left(H(t)\right)^{-1}G^*(t)-F(t), \quad (19)$$

and verifies the terminal condition $R(T)=C$.

Reciprocally, if $R(\cdot):(t_1,T]\to L\left(\mathbf{R}^n,\mathbf{R}^n\right)$ is the maximal solution of the Cauchy problem (19), with terminal condition $R(T)=C$, on the $L\left(\mathbf{R}^n,\mathbf{R}^n\right)$ and if $X(\cdot):(t_1,T]\to L\left(\mathbf{R}^n,\mathbf{R}^n\right)$ the solution of the Cauchy problem is:

$$\begin{cases}\dfrac{dX(t)}{dt}=\left(A(t)-B(t)\left(H(t)\right)^{-1}\left(G^*(t)-B^*(t)R(t)\right)\right)X(t),\\ X(T)=I_{\mathbf{R}^n}\end{cases} \tag{20}$$

and if $P(\cdot)$ is given by relation: $$P(t)=R(t)X(t),\ t\in(t_1,T], \tag{21}$$

then the application $\left(X(\cdot),P(\cdot)\right)$ is the matriceal solution of the Cauchy problem (6) – (7) and $X(t)\in L\left(\mathbf{R}^n,\mathbf{R}^n\right)$ is not singular for every $t\in(t_1,T]$ so that $t_1=t_0$, where t_0 is the one previously defined, and $(t_1,T]$ is the definition interval of the maximal solution $R(\cdot)$ of the Cauchy problem (18) with the terminal condition $R(T)=C$.

The theorem demonstration resumes, in essence, at the verification of the presuppositions and the consequences, which result.

Also, it is important to present the equivalent form, of the optimal synthesis $v(\cdot,\cdot)$ (from (11)) and the value function $W(\cdot,\cdot)$ (of (17)) in the new context, established by the 2. theorem.

So, we have: $$v(t,x)=-\left(H(t)\right)^{-1}\left(G^*(t)+B^*(t)R(t)\right)x \text{ and} \tag{22}$$

$$W(t,x)=\frac{1}{2}\left\langle R(t)x,x\right\rangle, \tag{23}$$

for any $(t,x)\in\tilde{E}=(t_1,T]\times\mathbf{R}^n$, where $R(\cdot):(t_1,T]\to L\left(\mathbf{R}^n,\mathbf{R}^n\right)$ is the maximal matriceal solution of the Riccati differential matriceal equation (19), which satisfies the terminal condition $R(T)=C$.

We also notice that this theorem practically establishes the equivalence between dynamic programming method and Riccati matriceal differential equation method (Kalman – Letov synthesis [9], [10]) for solving the quadric linear problem without restrictions. Comparing the obtained solutions (with both methods), which describe the optimal synthesis, we can make the following comments:

a). The utilization of the solution $\left(X(\cdot),P(\cdot)\right)$ of the problem (6) – (7) (Hamiltonian system with final conditions) for optimal synthesis expression by the (11) form presents the disadvantage of the huge dimension ($2n^2$) of the Hamiltonian system of differential equations, and the advantage that is a linear system;

b). The utilization of the solution $R(t)=P(t)\left(X(t)\right)^{-1},\ t\in(t_0,T]$ of the Riccati matriceal differential equation for optimal synthesis expression by the (22) equivalent form, presents the advantage of a small dimension (n^2), for the Riccati matriceal differential equation (19) but the disadvantage of the nonlinearity of these equations.

REFERENCES

1. GÂRBAN, Valentin – *Contributions at the Interaction Operators Study Into Multilevel Systems with Functional Hierarchy*, Bucharest, Military Technical Academy, 2000, Doctoral dissertation.
2. BERKOVITZ, L.D. - *Optimal Control Theory,* Springer, New York, 1974.
3. MIRICĂ Şt. – *Optimal Control. Sufficient Conditions and Synthesis.* Scientific Press, Bucharest, 1990.
4. CASTAING,C., VALADIER, M. – *Convex Analysis and Measurable Multifunctions.* Springer, New York, 1984.
5. ROCKAFELLAR, R. – *Convex Analysis.* Princetown University Press, Princetown, 1970.
6. HALANAY, A. – *Differential Equations,* Pedagogic and Didactic Press, Bucharest, 1972.
7. HARTMAN, P.H. – *Ordinary Differential Equations,* New York, 1964.
8. BELEA, C. – *System Theory. Non-Linear Systems,* Pedagogic and Didactic Press, Bucharest, 1985.
9. IONESCU, V. - *System Theory. Linear Systems,* Pedagogic and Didactic Press, Bucharest, 1985.
10. IONESCU, V., LUPAŞ, L.- *Calculus Techniques in System Theory*, vol I şi II, Technical Press, Bucharest 1974.
11. GÂRBAN, V. – *An general analysis and synthesis algorithm in optimal systems using dynamic programming method,* Military Technical Academy, Bucharest, 06-07 november, 2003, a XXX-a Sesiune de Comunicări Ştiinţifice cu participare internaţională, "Tehnologii Moderne în Secolul XXI".
12. GÂRBAN, V. – *Sisteme dinamice. Metode şi algoritmi în teoria sistemelor optimale,* Military Technical Academy Press, Bucharest, 2002.
13. GÂRBAN, V., UDREA, C. – *Ecuaţii diferenţiale şi ecuaţii cu derivate parţiale de ordinul I*, Editura Academiei Tehnice Militare, Bucureşti, 2001
14. ŞERBAN, S. - *Sisteme liniare,* Editura Printech, Bucureşti, 2002.
15. ŞERBAN, S. - *Sisteme dinamice liniare – aplicaţii numerice,* Editura Printech, Bucureşti, 2002.