RANDOM CASH-FLOWS AND RANDOM WALKS ON NETWORKS

Marin Danciu , *Institute for Public Administration,* Dublin, Irlanda
Executive Engineer Dun Laoghaire-Rathdown County Council

Dan Petru Vasiliu **,** UTM - Bucharest

INTRODUCTION : This paper summarizes some of the author's works concerning the sewerage and water network analysis . Mathematical concepts were developed in collaboration with Professor D.P.Vasiliu .
Random walks occurs in most problems in mathematical economics , providing basic models in forecasting problems.
Early development of the theory are related to issues in the stock market predictability .
For example, a random walk whose step size varies according to a gaussian distribution is the basic model for time series data such as financial markets (see the Black – Scholes formula in modelling option prices).
In many such models, a random – walk component is used to justify the unpredictability of the phenomenon.

A large enough number of economists involved in the field grant some degree of predicatbility,despite the existence of such a component .

For example, Andrew W. Lo and Archie Craig MacKinlay tried to prove this, considering the simple volatility – based test , ie :

$$\mathbf{X_t = \mu + X_{t-1} + e_t}$$

where:

$\mathbf{X_t}$ is the price of the stock at time t
μ is the trend parameter
e_t is a random disturbance , that is, even a random walk.

The aim of the paper is to study the random walk of two unities in a regular graph : this generates queueing and blocking phenomena that we study.
The two units can signify the supply and demand or even the evolution of two competitors on the same market .

KEYWORDS: regular graph ; poissonian queueing process ; random walk on a graph ; discrete random walk

Let $\mathbf{R = (V, A , v_0 , v_{n+1})}$ be a finite oriented network , having $\mathbf{V = \{ v_0, v_1, \ldots, v_{n+1} \}}$ as the set of verticies, $\mathbf{v_0}$ as the unique source and $\mathbf{v_{n+1}}$ the unique destination.
$\mathbf{A \subset V \times V}$ is the set of edges .Suppose $\mathbf{v_0}$ and $\mathbf{v_{n+1}}$ are unique determined with the property

$$\Gamma_0^{-1} = \Theta \ ; \Gamma_{n+1} = \Theta$$

where , for any vertex $v_i \in V$, we let

$$\Gamma_i = \Gamma(v_i) = \{ v_j \mid v_j \in V ; (v_i , v_j) \in A \}$$

$$\Gamma_i^{-1} = \Gamma^{-1}(v_i) = \{ v_j \mid v_j \in V ; (v_j , v_i) \in A \}$$

In addition, we assume that each vertex lies on at least one path connecting the source and the destination of the network.
We consider the case of two mobile units, which initially lies in the source , then randomly moving to its destination . Let us define the functions f, g, as follows

$$q : A \to R^{+} \ ; \mu : V \to V \setminus \{ v_{n+1} \} \to R^{+}$$

To denote the values of these functions we use the next abbreviations

$$q(v_i , v_j) = q_{ij} \ , for \ (v_i , v_j) \in A \ ; \mu (v_i) = \mu_i \ , for \ v_i \in V \setminus \{ v_{n+1} \}$$

Suppose $\mathbf{q}$ fits the condition

$$\sum_{(v_i , v_j) \in A} q_{ij} = 1, for\ every \quad i = \overline{0;n}$$

The values of μ plays the role of stationary parameters of the units in the vertexes of $\mathbf{R}$, according to the Poisson hypotheses .
The two units certainly lies in the source at time t = 0 :
the random variable which gives the distribution of the number of units that get out of the source in time interval (0, t) is

$$X = \begin{pmatrix} 0 & 1 & 2 \\ \exp(-\mu_0 \cdot t) & \mu_0 \cdot t \cdot \exp(-\mu_0 \cdot t) & 1 - \exp(-\mu_0 \cdot t) - \mu_0 \cdot t \cdot \exp(-\mu_0 \cdot t \end{pmatrix} .$$

A unit located at node $v_i \in V$ at time $\mathbf{t}$ leaves this node during the time interval $(t ; t + \Delta t)$ with probability $\mu_i \cdot \Delta t + 0(\Delta t)$ and reaches a node $\mathbf{v_j}$ with probability $\mathbf{q_{ij}}$ going instantaneously through the arc $(v_i , v_j) \in A$. The unit is absobed at $\mathbf{v_{n+1}}$.

We denote by $\mathbf{p_{ij}(t)};\mathbf{i,j}=\overline{0;\mathbf{n}+1}$ the probability that at time t one of the units be in the vertex $\mathbf{v_i}$ and the other one in the vertex $\mathbf{v_j}$. These probabilities are called position probabilities and are supposed to verify the relationship

$$\mathbf{p_{ij}(t)}=\mathbf{p_{ji}(t)};\forall\,\mathbf{t}\geq 0;\forall\mathbf{i},\mathbf{j}=\overline{0;\mathbf{n}+1}$$

Of a particular interest are hypotheses that can be made if the two units meet in a point different from source and destination. In the next we'll analyze two of these

A: when the two units meet together in the vertex $\mathbf{v_i}$ (different from the source and the destination) a queueing phenomenon occurs : this phenomenon follows an exponential law of parameter $\boldsymbol{\mu_i}$

B: when the two units meet together in the vertex different from the source and the destination, the units remain locked in that node and the random walks stops

In this paper only the first option is analysed

We'll extend the values of the function $\mathbf{q}$ to the whole $\mathbf{A}$ by putting

$$\mathbf{q}(\mathbf{v_i},\mathbf{v_j})=0 \ \mathbf{if}\,(\mathbf{v_i},\mathbf{v_j})\notin\mathbf{A}.$$

let also $\boldsymbol{\mu_{n+1}} = \mathbf{0}$ **;** then the position probabilities verify the system of differential equations below

$$(1):\mathbf{p_{00}(t)}=\mathbf{exp}(-\mu_0\cdot\mathbf{t})$$

$$(2):\mathbf{p'_{0j}(t)}+(\mu_0+\mu_j)\cdot\mathbf{p_{0j}(t)}=\sum_{\mathbf{k}=0}^{\mathbf{n+1}}\mathbf{p_{0k}(t)}\cdot\mu_k\cdot\mathbf{q_{kj}}$$

$$\mathbf{j}=\overline{1;\mathbf{n}+1}$$

$$(3):\mathbf{p'_{ij}(t)}+(\mu_i+\mu_j)\cdot\mathbf{p_{ij}(t)}=\sum_{\mathbf{k}=0}^{\mathbf{n+1}}\mathbf{p_{kj}(t)}\cdot\mu_k\cdot\mathbf{q_{ki}}+\sum_{\mathbf{k}=0}^{\mathbf{n+1}}\mathbf{p_{ik}(t)}\cdot\mu_k\cdot\mathbf{q_{kj}},$$

$$\mathbf{i},\mathbf{j}=\overline{1;\mathbf{n}+1};\mathbf{i}<\mathbf{j}$$

$$(4):\mathbf{p'_{ii}(t)}+\mu_i\cdot\mathbf{p_{ii}(t)}=\sum_{\mathbf{k}=0}^{\mathbf{n+1}}\mathbf{p_{ki}(t)}\cdot\mu_k\cdot\mathbf{q_{ki}}\ ;\mathbf{i}=\overline{1;\mathbf{n}+1}$$

here, (1) comes out of the definition of $\mathbf{X}$, and (2) results from the following

$$p_{0j}(t+\Delta t)=p_{0j}(t)\cdot(1-\mu_j\cdot\Delta t)+\sum_{\substack{v_k\in\Gamma_j^{-1}\\ k\neq 0;k\neq j}} p_{0k}(t)\cdot\mu_k\cdot q_{kj}\cdot\Delta t\cdot(1-\mu_0\cdot\Delta t)+$$
$$+p_{00}(t)\cdot\mu_0\cdot q_{0j}\cdot\Delta t+O(\Delta t),\ j\neq 0$$

hence, for $\Delta t\to 0$, we get;

$$p_{0j}^{'}(t)=-(\mu_0+\mu_j)\cdot p_{0j}(t)+\sum_{\substack{v_k\in\Gamma_j^{-1}\\ k\neq 0;k\neq j}} p_{0k}(t)\cdot\mu_k\cdot q_{kj}+p_{00}(t)\cdot\mu_0\cdot p_{0j}\ ;j\neq 0$$

By the convention that $q_{ij}=0$ **if** $(v_i,v_j)\notin A$, the previous relation takes the shape of (2)

For further development we'll deal with

$$\begin{aligned} p_{ij}(t+\Delta t)=&\,p_{ij}(t)\cdot(1-\mu_i\cdot\Delta t)\cdot(1-\mu_j\cdot\Delta t)+\\ &+\sum_{\substack{v_k\in\Gamma_i^{-1}\\ k\neq i;k\neq j}} p_{kj}(t)\cdot\mu_k\cdot q_{ki}\cdot\Delta t\cdot(1-\mu_j\cdot\Delta t)+\\ &+\sum_{\substack{v_k\in\Gamma_j^{-1}\\ k\neq i;k\neq j}} p_{ik}(t)\cdot\mu_k\cdot q_{kj}\cdot\Delta t\cdot(1-\mu_i\cdot\Delta t)+\\ &+p_{ii}(t)\cdot\mu_i\cdot q_{ij}\cdot\Delta t+p_{jj}(t)\cdot\mu_j\cdot q_{ji}\cdot\Delta t+O(\Delta t)\ ,\ i<j;i\neq 0 \end{aligned}$$

respectively

$$p_{ii}(t+\Delta t)=p_{ii}(t)\cdot(1-\mu_i\cdot\Delta t)+\sum_{\substack{v_k\in\Gamma_i^{-1}\\ k\neq i}} p_{ki}(t)\cdot\mu_k\cdot q_{ki}\cdot\Delta t\cdot(1-\mu_i\cdot\Delta t)+O(\Delta t),i\neq 0$$

BIBLIOGRAPHY:

1: L.Lovasz : Random walks on graphs: a survey , Keszthely (Hungary) 1993
2: P.Tetaly :Random walks and effective resistance of networks,
in : J. of Theoretical probability 1 (1991) 101-109
3: D.P.Vasiliu : Mathematical methods for the economic decision , Editura Tehnica,
Bucuresti, 1986